XXIV. *Some Properties of the Sum of the Divisors of Numbers.* *By* Edward Waring, *M. D. F. R. S.*

Read May 29, 1788.

1. LET the equation $\overline{x-1}\,.\,\overline{x^2-1}\,.\,\overline{x^3-1}\,.\,\overline{x^4-1}\,.\,\overline{x^5-1}\,.\ldots\overline{x^n-1} = x^b - px^{b-1} + qx^{b-2} - rx^{b-3} + sx^{b-4} - \&c. = x^b - x^{b-1} - x^{b-2} + x^{b-5} + x^{b-7} - x^{b-12} - x^{b-15} + x^{b-22} + x^{b-26} - x^{b-35} - x^{b-40} + x^{b-51} + x^{b-57} - \&c. \ldots x^{b-n} \pm \&c. = A = 0$. The signs + and − proceed alternately by pairs unto the term x^{b-n}. The co-efficients of all the terms to the above-mentioned (x^{b-n}) will be $+1$, -1 or 0; they will be $+1$, when multiplied into x^{b-v}, where $v = \frac{3z^2+z}{2}$ or $= \frac{3z^2-z}{2}$, and z an even number; but -1, if z be an uneven number; in all other cases they will be $= 0$.

The numbers 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, &c. subtracted from b may be collected from the addition of the numbers 1, 1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, &c. which consist of two arithmetical serieses 1, 3, 5, 7, 9, 11, &c. 1, 2, 3, 4, 5, 6, 7, &c. intermixed.

2. The sum of any power (m) of each of the roots in the equation $A = 0$ will be $S(m)$, where $S(m)$ denotes the sum of all the divisors of the number m, if m be not greater than n.

Cor. Hence (by the rule for finding the sum of (m) powers of each of the roots from the sum of the inferior powers and co-efficients of the given equation) may be deduced $S(m) =$

$pS(m-1) - qS(m-2) + rS(m-3) - sS(m-4) + tS(m-5) -$ &c. $= S(m-1) + S(m-2) - S(m-5) - S(m-7) + S(m-12) + S(m-15) - S(m-22) - S(m-26) +$ &c. which is the property of the ſum of diviſors invented by the late M. EULER.

Cor. By ſubſtituting for $S(m-1)$, $S(m-2)$, &c. their values $S(m-2) + S(m-3) - S(m-6) - S(m-8) +$ &c., $S(m-3) + S(m-4) - S(m-7) - S(m-9) +$ &c. &c. in the given equation $S(m) = S(m-1) + S(m-2) - S(m-5) - S(m-7) +$ &c. may be acquired an expreſſion for the ſum $S(m)$ in terms of the ſums of the diviſors of numbers leſs than $m-1$, $m-2$, &c. – the ſame method may be uſed for a ſimilar purpoſe in ſome of the following propoſitions.

Cor. By the rule for finding the ſum of the contents of every (m) roots from the ſums of the powers of each of the roots may be deduced the equation $\pm 1 . 2 . 3 . 4 \ldots m$, or $o = 1 - m . \frac{m-1}{2} S(2) + m . \overline{m-1} . \frac{m-2}{3} S(3)$

$$- m . \overline{m-1} . \overline{m-2} . \frac{m-3}{4} S(4) + \text{\&c.}$$

$$+ m . \overline{m-1} . \frac{m-2}{2} . \frac{m-3}{2^2} S((2))^2 - \text{\&c.}$$

in which the ſum of the diviſors of any number m is expreſſed by the ſums of the diviſors of the inferior numbers $m-1$, $m-2$, &c. and their powers. If v be an even number, then $\pm 1 . 2 . 3 \ldots m$ will have the ſame ſign as the co efficient; if uneven, the contrary; but if the co-efficient $= 0$, then will the content $1 . 2 . 3 \ldots m$ vaniſh. The law of this ſeries is given in the Meditationes Algebraicæ.

3. Let H be the number of different ways by which the ſum of any two numbers $1, 2, 3, 4, \ldots m-2, m-1$, can become $= m$; H' the number of ways by which the ſum of any three of the above-mentioned numbers can make m; H'', H''', H'''', &c. the number of ways by which the ſum of any four, five, ſix,

 &c.

&c. of the above-mentioned numbers is $= m$ respectively; then will $1 - H + H' - H'' + H''' - \&c. = \pm 1$ or 0. Let $m = \frac{3z^2 \pm z}{2}$, and it will be $+1$ or -1, according as z is an odd or even number, in all other cases it will be $= 0$.

PART II.

1. Let the equation be $\overline{x-1} . \overline{x^2-1} . \overline{x^3-1} . \overline{x^5-1} . \overline{x^7-1} . \overline{x^{11}-1} . \overline{x^{13}-1} . \overline{x^{17}-1} \ldots \overline{x^n-1}$. &c. $= x^{b'} - px^{b'-1} + qx^{b'-2} - rx^{b'-3} + sx^{b'-4} - \&c. = x^{b'} - x^{b'-1} - x^{b'-2} + x^{b'-4} + x^{b'-8} - x^{b'-10} - x^{b'-11} + x^{b'-12} + x^{b'-16} - x^{b'-17} - x^{b'-19} + x^{b'-20} - x^{b'-23} + 2x^{b'-24} - x^{b'-26} - x^{b'-27} + x^{b'-28}$, &c. $= A' = 0$; the sum of any power (m) of each of the roots in the equation $A' = 0$ will be $S'(m)$, where $S'(m)$ denotes the sum of all the prime divisors of the number m, and m is not greater than n.

Cor. Hence, by the rule before-mentioned $S'(m) = S'(m-1) + S'(m-2) - S'(m-4) - S'(m-8) + S'(m-10) + S'(m-11) - S'(m-12) - S'(m-16) + S'(m-17) + S'(m-19) - S'(m-20) + S'(m-23) - 2S'(m-24) + S'(m-26) + S'(m-27) - S'(m-28) + S'(m-29)$, &c.

If in this, or the preceding, or subsequent analogous cases $S(m-r)$, or $S'(m-r)$, or $S^{l}(m-r)$, becomes $S(0)$, or $S'(0)$, or $S^{l}(0)$; for $S(0)$, or $S'(0)$, or $S^{l}(0)$, always substitute r.

Cor. Let L be the co-efficient of the term $x^{b'-m}$; then, by the above-mentioned series contained in the Meditationes Algebraicæ, will $1 . 2 . 3 . 4 \ldots m \times L = 1 - m . \frac{m-1}{2} S'(2) + m . \overline{m-1} . \frac{m-2}{3} \times S'(3) - m . \overline{m-1} . \overline{m-2} . \frac{m-3}{4} \times S'(4)$
+ &c. $\qquad + m . \overline{m-1} . \overline{m-2} . \frac{m-3}{8} \times S'((2))^2$
$-$ &c.

3

– &c. be an equation, which expresses a relation between the prime divisors of the numbers 1, 2, 3, 4 . . . $m-1$, m, and their powers.

Cor. The co-efficient L = the difference between the two respective numbers of different ways that m can be formed adding the prime numbers 1, 2, 3, 5, 7, 11, 13, 19, &c. the one with, and the other without, 2.

PART III.

1. Let an equation $\overline{x^{\alpha}-1}\,.\,\overline{x^{\beta}-1}\,.\,\overline{x^{\gamma}-1}\,.\,\overline{x^{\delta}-1}\times\&c. = x^{b}-px^{b-1}+qx^{b-2}-rx^{b-3}+\&c.=0$; then will the sum of the (m) powers of each of its roots be the sum of all the divisors of m, that can be found amongst the numbers α, β, γ, δ, &c.

2. The co-efficient of the term x^{b-m} will be the difference between the two respective numbers of different ways, that the number (m) can be formed from the addition of the numbers α, β, γ, δ, &c.; the one containing in it an odd number of the even numbers contained in α, β, γ, δ, &c.; the other not.

PART IV.

1. Let $\overline{x^{l}-1}\,.\,\overline{x^{2l}-1}\,.\,\overline{x^{3l}-1}\,.\,\overline{x^{4l}-1}\,.\,.\,.\,.\,.\,\overline{x^{nl}-1}\,.\,\&c. = x^{b}-px^{b-l}+qx^{b-2l}-rx^{b-3l}+\&c. = x^{b}-x^{b-l}-x^{b-2l}+x^{b-5l}+x^{b-7l}-x^{b-12l}-x^{b-15l}+\&c. = B = 0$, of which equation all the co-efficients are the same as in case the first, and consequently ± 1 or 0 to the term (x^{b-nl}).

2. The sum of any power $l\times m$ of each of the roots of the equation $B=0$ will be $S^{l}(m)$, where $S^{l}(m)$ denotes the sum of the divisors of m, which are divisible by l.

Cor.

Cor. Hence $S^l(m) = S^l(m-l) + S^l(m-2l) - S^l(m-5l) - S^l(m-7l) + S^l(m-12l) + S^l(m-15l) - S^l(m-22l) - S^l(m-26l) +$ &c.; the law of the series has been given in Case 1.

Cor. The sum of all the divisors of m not divisible by $l = S(m) - S^l(m) = S(m-1) - S^l(m-l) + (S(m-2) - S^l(m-2l)) - (S(m-5) - S^l(m-5l)) - (S(m-7) - S^l(m-7l)) +$ &c.

A similar rule may be predicated of the sum of the divisors not divisible by the numbers a, b, c, d, &c: for the sum of the divisors of the number (m) divisible by a, b, c, d, e, &c., where a, b, c, d, e, &c. are prime to each other $= (S^a(m) + S^b(m) + S^c(m) + S^d(m) + S^e(m) +$ &c.$)) - ((S^{a\times b}(m) + S^{a\times c}(m) + S^{b\times c}(m) + S^{a\times d}(m) + S^{b\times d}(m) + S^{c\times d}(m) + S^{a\times e}(m) +$ &c$)) + (S^{a\times b\times c}(m) + S^{a\times b\times d}(m) + S^{a\times b\times d}(m) + S^{b\times c\times d}(m) + S^{a+b+}(m) +$ &c.$)) - ((S^{a+b+c+d}(m) + S^{a+b+c+e}(m) +$ &c.$)) + (S^{a+b+c+d+}(m) +$ &c.$)) -$ &c. $= l =$ the sum of all the divisors of m .... divisible by a, b, c, d, e, &c. respectively added together, $-$ the sum of all the divisors of m divisible by the products (ab, ac, bc, &c.) of any two of the quantities a, b, c, d, &c. $+$ the sum of all the divisors of m divisible by the contents (abc, abd, acd, bcd, &c.) of every three of the quantities a, b, c, d, &c. $-$ the sum of all the divisors of m divisible by the contents of every four of the abovementioned quantities a, b, c, d, &c. $+$ and so on, and consequently $S(m) - C$ is the sum required.

The principles given in the former parts may be applied to this, and extended to equations of which the factors have the formula $x^a \pm k$; and from the sum of the inferior powers of each of the roots, and the co-efficients may be collected the sum of the superior; the same may be performed by the co-efficients only, &c.

PART

PART V.

1. $S(\alpha\times\beta)=\alpha\times S(\beta)+$ ſum of all the diviſors of β not diviſible by $\alpha=\beta\times S(\alpha)+$ ſum of all the diviſors of α not diviſible by β.

2. $S'(\alpha\times\beta)=\alpha\times S'(\beta)+$ ſum of all the diviſors of β diviſible by l but not by $\alpha=$ &c.

3. $S(\alpha\times\beta\times\gamma\times\delta\times$ &c.$)=\alpha\times S(\beta\times\gamma\times\delta\times\varepsilon$, &c.$)+$ ſum of all the diviſors of $\beta\times\gamma\times\delta\times\varepsilon$, &c. not diviſible by $\alpha=\alpha\times\beta\times S(\gamma\times\delta\times\varepsilon$, &c.$)+$ ſum of all the diviſors of $\beta\times\gamma\times\delta\times\varepsilon$, &c. not diviſible by $\alpha+\alpha\times$ ſum of all the diviſors of $\gamma\times\delta\times\varepsilon$, &c. not diviſible by $\beta=\alpha\times\beta\times\gamma\times S(\delta\times\varepsilon$, &c.$)+$ ſum of all the diviſors of $\beta\times\gamma\times\delta\times\varepsilon$, &c. not diviſible by $\alpha+\alpha\times$ ſum of all the diviſors of $\gamma\times\delta\times\varepsilon$, &c. not diviſible by $\beta+\alpha\times\beta\times$ ſum of all the diviſors of $\delta\times\varepsilon$, &c. not diviſible by $\gamma=\alpha\times\beta\times\gamma\times\delta\times S(\varepsilon$, &c.$)+$ ſum of all the diviſors of $\beta\times\gamma\times\delta\times\varepsilon$, &c. not diviſible by $\alpha+\alpha\times$ ſum of all the diviſors of $\gamma\delta\varepsilon$, &c. not diviſible by $\beta+\alpha\times\beta\times$ ſum of all the diviſors of $\delta\varepsilon$, &c not diviſible by $\gamma+\alpha\times\beta\times\gamma\times$ ſum of all the diviſors of ε, &c. not diviſible by $\delta=$ &c. The law of the ſeries is manifeſt. The letters α, β, γ, δ, &c. which are not contained between the parentheſes, denote prime numbers.

Cor. If ſome of the letters α, β, γ, δ, &c. be ſubſtituted for others, and others for them, the equations reſulting will be juſt, and conſequently many new equations may be deduced.

If in the preceding equations for S be wrote S′, and for the ſum of all the diviſors of a certain quantity not diviſible by a prime number (α, or β, or γ, &c.) be wrote the ſum of all the

the diviſors of that quantity not diviſible by the ſame prime number, but diviſible by l; the propoſitions reſulting will be true.

Theſe equations may be applied to the equations given in the preceding parts, and from thence many others deduced.